Bayesian model-independent evaluation of expansion rates of the universe

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Abstract Marginal likelihoods for the cosmic expansion rates are evaluated using the 'Constitution' data of 397 supernovas, thereby updating the results in some previous works. Even when beginning with a very strong prior probability that favors an accelerated expansion, we obtain a marginal likelihood for the deceleration parameter q_0 peaked around zero in the spatially flat case. It is also found that the new data significantly constrains the cosmographic expansion rates, when compared to the previous analyses. These results may strongly depend on the Gaussian prior probability distribution chosen for the Hubble parameter represented by h, with $h = 0.68 \pm 0.06$. This and similar priors for other expansion rates were deduced from previous data. Here again we perform the Bayesian model-independent analysis in which the scale factor is expanded into a Taylor series in time about the present epoch. Unlike such Taylor expansions in terms of redshift, this approach has no convergence problem.

Keywords Cosmography; SN Ia data; Cosmic expansion rates; Deceleration parameter; marginal likelihoods

1 Introduction

It is generally accepted that a more appropriate way to measure the acceleration of expansion of the universe is to resort to a cosmographic or model-independent analysis. In the conventional model-based analyses of distance modulus-redshift ($\mu - z$) data of Type Ia supernova (SN Ia), the accelerated expansion of the universe is an indirect inference based on the best fit values of parameters, such as the density parameters Ω_m, Ω_Λ, etc.

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On the other hand, in a model-independent approach, the scale factor $a(t)$ is expanded as a Taylor series in time about the present epoch and the marginal likelihoods of its coefficients are computed using the data. The marginal likelihood for the deceleration parameter gives an estimate of the acceleration of cosmic expansion. Since practically one has to truncate the series to some finite order, the basic assumption here is that $a(t)$ is expressible as a truncated Taylor series or polynomial. Evaluating the deceleration parameter by adopting this method, it was confirmed model-independently that the universe is undergoing an accelerated expansion (John 2004, 2005).

In this paper we report the updating of the marginal likelihood for each of the expansion coefficients found in the above work. This is performed for the case of a fifth order polynomial. A notable result in the present Bayesian model-independent analysis is that even when beginning with a very strong prior probability that favors an accelerated expansion, the marginal likelihood for the deceleration parameter q_0 is found peaked around $q_0 = 0$ in the spatially flat case. It is also found that the new data significantly constrains the cosmic expansion rates appearing in the Taylor expansion, when compared to the previous data. We also note that successive terms in the series decreases sufficiently fast, thereby verifying the assumption of a converging Taylor series in time for the cosmic scale factor.

Other model-independent approaches, which Taylor expand the distance modulus μ in terms of redshift z, have also gained attention in recent years [See for eg. (Shapiro & Turner 2006; Cattoen & Visser 2007; Guimaraes et al. 2009; Seikel & Schwarz 2009)]. But a drawback of this method is that, in principle, it converges only for $\mid z \mid < 1$ (Cattoen & Visser 2007; Guimaraes et al. 2009). The argument behind this assertion is as follows: For an expanding universe, $z < 0$ corresponds to the future and $z = -1$ is the redshift

when the universe has expanded to infinite size. Since $z = -1$ is a pole, by standard complex variable theory, the radius of convergence of a series about $z = 0$ is atmost $\mid z \mid = 1$, so that it fails to converge for $z > 1$. When compared to this, our approach of expanding the scale factor in terms of t about the present epoch t_0 is advantageous, for the series converges for all times. Even the lookback time $T \equiv t - t_0$ is evaluated by numerically solving an equation which involves a Taylor series in time. Hence there is no convergence problem in the present work. However, it should be noted that all analyses which make use of such Taylor expansions, in practice, employ polynomials and hence convergence is not a serious problem. For instance, one can see that there is convergence in certain special cases of the low order polynomial fit by Guimaraes et al. (2009).

2 Marginal likelihoods for the cosmic expansion rates

With $t - t_0 \equiv T$, where t_0 is the present time, the scale factor of the universe is expanded into a Taylor series about the present epoch t_0 as (John 2004, 2005)

$$a(t_0 + T) = a_0 \times$$
$$\left[1 + H_0 T - \frac{q_0 H_0^2}{2!} T^2 + \frac{r_0 H_0^3}{3!} T^3 - \frac{s_0 H_0^4}{4!} T^4 + \frac{u_0 H_0^5}{5!} T^5 + .. \right] \quad (1)$$

Note that T assumes negative values for the past. For a $k = 0$ flat universe, one can put $a_0 = 1$. The remaining parameters in the theory are the Hubble parameter $H_0 \equiv 100h$ km s^{-1} Mpc-1 , the deceleration parameter q_0 and higher order expansion rates such as r_0, s_0, u_0, etc. Our task is to deduce the values of these parameters from the observational data.

For a light pulse emitted from a SN situated at the coordinate r_1 at time t_1 and reaching us at $r = 0$ at time t_0, the RW metric allows one to write

$$\int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{r_1}^{0} \frac{dr}{(1-kr^2)^{1/2}}. \quad (2)$$

For a $k = 0$ RW metric, this can be used to obtain

$$r_1 = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{T_1}^{0} \frac{dT}{a(t_0 + T)}. \quad (3)$$

With this, we can compute the luminosity distance $D = r_1 a_0 (1 + z)$. An important part of the calculation is the solution of the equation

$$1 + z = \frac{a(t_0)}{a(t_0 + T_1)}, \quad (4)$$

used to find T_1 in terms of z, for each combination of parameter values. This is done in a direct and purely numerical way.

We may now obtain the distance modulus as $\mu = 5 \log (D/1\mathrm{Mpc}) + 25$. Here D and hence μ are functions of z and contain parameters h, q_0, r_0, s_0, u_0, etc. In the following, we keep only terms up to fifth order in the Taylor expansion.

The likelihood function is

$$\mathcal{L} = \exp[-\chi^2(h, q_0, r_0, s_0, u_0)/2]$$

where χ^2 is given by

$$\chi^2 = \Sigma_k \left(\frac{\hat{\mu}_k - \mu_k(z_k; h, q_0, r_0..)}{\sigma_k} \right)^2. \quad (5)$$

Here $\hat{\mu}_k$ is the measured value of the distance modulus of the kth supernova, $\mu_k(z_k; h, q_0, ..)$ is its expected value (from theory) and σ_k is the uncertainty in the measurement.

The likelihood for the truncated Taylor series form of scale factor can be found as

$$\mathcal{L}(M) = \int dh \int dq_0 \int dr_0 \int ds_0 \int du_0$$
$$p(h)p(q_0)p(r_0)p(s_0)p(u_0) \, e^{-\chi^2/2}. \quad (6)$$

where $p(h)p(q_0)p(r_0)p(s_0)p(u_0)$ is a product of Gaussian probability distributions of each of the parameters. This is an approximation to $p(h, q_0, r_0.....)$, the prior probability distribution to be introduced in equation (6).

The marginal likelihood of any one parameter can be computed by integrating the likelihood function, multiplied by an appropriate prior probability distribution, over all parameters except the concerned one. For instance, the marginal likelihood for q_0 can be found as

$$\mathcal{L}(q_0) = \int dh \int dr_0 \int ds_0 \int du_0$$
$$p(h)p(r_0)p(s_0)p(u_0) \, e^{-\chi^2/2}. \quad (7)$$

A salient feature in the present computation is that while using (7), the marginal likelihoods obtained in the previous analysis (John 2004, 2005) are taken as the prior probability distributions, for the corresponding coefficients. These references have used flat priors, since there were no other previous work evaluating these marginal likelihoods. But it was proposed there itself that the posterior marginal likelihoods obtained in it shall be used as priors for future analyses. We note that the present work is the appropriate place to make

use of this. However, it would not be computationally feasible to use the posterior in the previous analysis as prior, in terms of a table of values. Therefore, we approximate those distributions by Gaussian functions, with the corresponding mean and standard deviations obtained in (John 2005). A comparison with the actual plots show that this is a reasonable approximation for most coefficients. The product of such individual priors is the combined prior, used in equation (7).

It should be noted that the ranges of flat priors in the previous analysis were chosen on the basis of the same 'All SCP' SN data in Knop et. al. (2003) itself, eventhough it ran the risk of using the same data twice. There we first found the ranges of the 'contributing' values of the parameters by varying them arbitrarily. Later, flat prior probabilities were assigned for those ranges and they were used to find the posterior marginal likelihoods. In the present analysis, the marginal likelihoods thus obtained are used as priors.

As mentioned above, the present model-independent analysis uses the 'Constitution' data (Hicken et. al. 2009) of 397 SN. In this connection, it shall be noted that some SN Ia, which appeared in the dataset of 54 'All SCP' SN used in the previous analysis are present in the Constitution set too. But the values of μ_k, z_k and the errorbars of such objects are found modified to some extent in the new release. Therefore, it was opted not to exclude such SN from the Constitution set. We note that at any rate, the Gaussian priors obtained from the previous analysis are a better option than flat priors.

We have computed the marginal likelihoods of four important expansion rates of the present universe, namely q_0, r_0, s_0 and u_0, and the results are shown in Figs. (1)-(4). Terms up to fifth order are kept in the expansion, but only the flat ($k = 0$) case is considered. This is equivalent to assuming a δ-function prior for the flat spatial geometry. The joint prior probability used for other parameters was the product of individual Gaussian functions in each parameter with mean and standard deviations as follows: $h = 0.68 \pm 0.06$, $q_0 = -0.90 \pm 0.65$, $r_0 = 2.7 \pm 6.7$, $s_0 = 36.5 \pm 52.9$, and $u_0 = 142.7 \pm 320$ (John 2005). In each case, the integrations were performed in the 2σ range of each of the parameters. We have performed variation with respect to h, though marginal likelihood for this parameter was not drawn. The step sizes chosen for these parameters were $\Delta h = 0.01$, $\Delta q_0 = 0.1$, $\Delta r_0 = 1$, $\Delta s_0 = 20$ and $\Delta u_0 = 100$.

The results show that there is significant constraining of the parameters while using the new and refined data, compared to the corresponding results in (John 2005). It is to be reminded that the marginal likelihoods are not precisely probability distributions for the
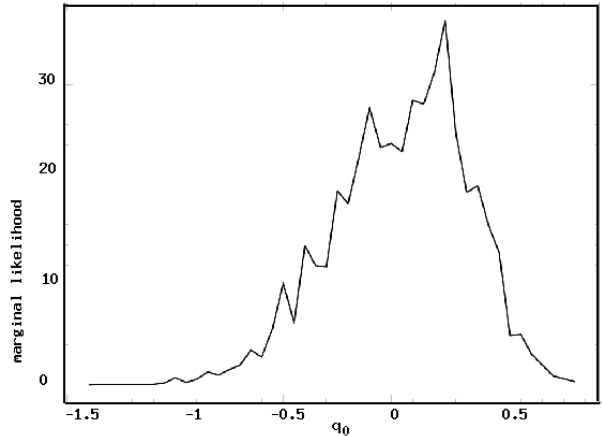

Fig. 1 Marginal likelihood for the parameter q_0 (in units of 10^{-105}), while using the polynomial of order 5
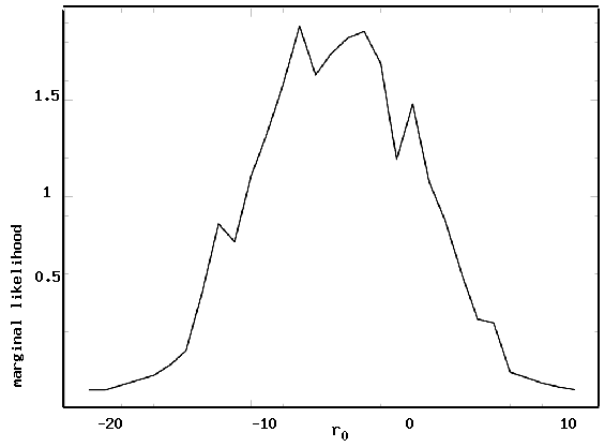

Fig. 2 Marginal likelihood for the parameter r_0 (in units of 10^{-105}), while using the polynomial of order 5

parameters; instead, they are the probability for the data, given the model and the parameter values. However, we here compute mean and standard deviations, considering the marginal likelihoods as distributions. The new mean and standard deviations are the following: $q_0 = 0.04 \pm 0.30$, $r_0 = -4.5 \pm 4.6$, $s_0 = -42.8 \pm 52.5$, and $u_0 = 320.5 \pm 213.0$. The marginal likelihood for q_0 obtained by John (2005), which is also used as prior for this parameter in the present work, is reproduced here in Fig. (5) for comparison. The standard deviations of each of these parameters, except that of s_0, have decreased substantially and this leads to the above assertion that the Constitution data constrains the cosmic expansion rates significantly.

It shall be noted that even when beginning with a prior probability distribution centred around $q_0 = -0.9$, which is strongly in favor of an accelerated ex-
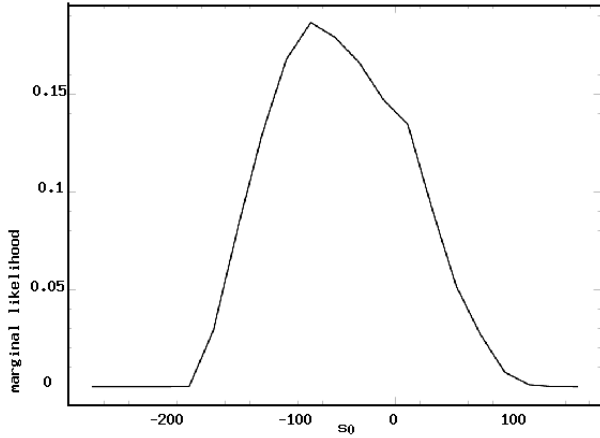
Fig. 3 Marginal likelihood for the parameter s_0 (in units of 10^{-105}), while using the polynomial of order 5
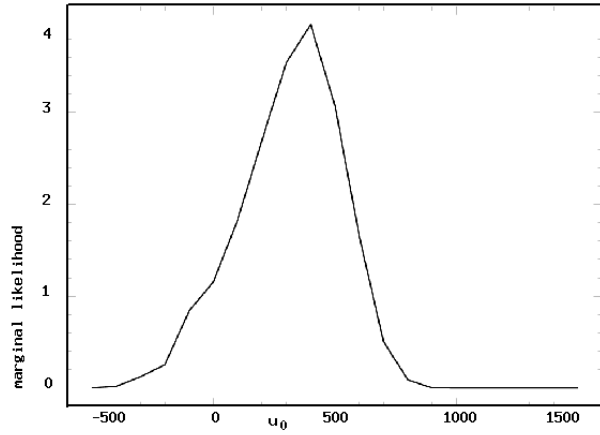

Fig. 4 Marginal likelihood for the parameter u_0 (in units of 10^{-107}), while using the polynomial of order 5

pansion, we ended up with a marginal likelihood peaked around $q_0 \approx 0$. Thus whereas the data in (Knop et. al. 2003) validated the claim of accelerated expansion, the Constitution SN dataset in Hicken et. al. (2009) favors a coasting evolution; i.e., the universe may neither be accelerating nor decelerating. However, the presence of substantial amount of dark energy and dark matter would still be required to explain the data.

Here one observes that the result $q_0 \approx 0$ could be connected to the value of h and also that properly including h in the analysis may further decrease the constraining power of the data. Therefore one should explore the consequences of using Gaussian priors on h from other measurements too. In fact, we have marginalised the likelihoods over the Hubble parameter, with a Gaussian prior $h = 0.68 \pm 0.06$, as mentioned above. But since the likelihood curve for h obtained
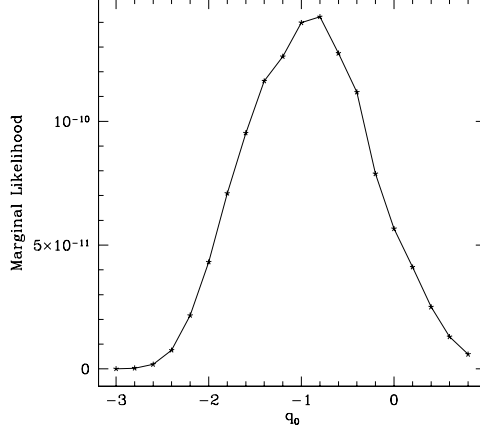

Fig. 5 Marginal likelihood for the parameter q_0, obtained in John (2005), while using the polynomial of order 5 and the data in Knop et. al. (2003).

from the previous analysis by John (2005) is not very sensitive to its variation in the concerned range (unlike the case of q_0, r_0, etc.), it is more appropriate to employ priors for h deduced from other measurements. We propose that this procedure shall be followed in future analyses.

The considerable spread left in the marginal likelihoods shows that even now there is some freedom in choosing the values of those parameters. In other words, there is a sizable volume in the parameter space that can have the same low χ^2. But this should not be viewed as a drawback of the analysis; instead, this simply reflects the fact that the data are not yet accurate enough. Some recent analyses of Constitution SN data endorses this result (Shafieloo et.al. 2009). This freedom in SN data was noted earlier (John 2004, 2005), which highlights the strength of the Bayesian model-independent approach.

Based on the mean values obtained for these parameters, we compute the successive terms in the series (2). With time in units of 10^{17} s, the series can be written as

$$1 + 2.106 \times 10^{-1}T - 2.22 \times 10^{-2}q_0T^2$$
$$+1.55 \times 10^{-3}r_0T^3 - 0.819 \times 10^{-4}s_0T^4$$
$$+3.45 \times 10^{-6}u_0T^5 + ...,$$

where we have taken $h = 0.65$ (only to evaluate this series). With the values of the parameter in the ranges

obtained in the analysis, this series appears to converge even for $\mid T \mid$ as large as $\approx 3 \times 10^{17}$ s. However, this feature is not essential for our analysis, for we have assumed only a polynomial form for the scale factor. The situation was not different in the previous work either.

3 Conclusion

We assumed that a Taylor series form for the scale factor $a(t)$ is valid and attempted to find the coefficients in this expansion using the recent Constitution SN data. The new marginal likelihoods obtained for its coefficients give valuable information regarding the expansion history of the universe. It is found that there is significant constraining of these parameters when compared to previous analyses using the data in (Knop et. al. 2003). The shift in the computed mean value of the deceleration parameter q_0, from that found in the previous analysis is noteworthy. Even when we start with a prior probability distribution that strongly favors an accelerating universe, the marginal likelihood for the deceleration parameter obtained from the present analysis using the Constitution data is found peaked around $q_0 = 0$. However, we re-iterate that the considerable spread still found in the likelihoods of these parameters indicate freedom in the choice of their numerical values.

A distinguishing feature of our analysis is that the marginal likelihoods for each parameter obtained in the previous case is chosen as the prior probability distribution in the present one, thereby implementing the Bayesian method in true spirits. The work is also intended as a demonstration of this fundamental requirement in Bayesian analysis. However, we have noted that the results obtained in this paper may heavily depend on the prior chosen for h. Thus it is important to evaluate expansion rates using prior for h deduced from other measurements too. It is expected that in future when the SN dataset becomes large enough, the expansion coefficients get sharply peaked marginal likelihoods and become the most basic model-independent description of the expansion history of the universe.

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